On L1 compactness in transport theory

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Abstract

We give a systematic and nearly optimal treatment of the compactness in connection with the L^1 spectral theory of neutron transport equations on both n-dimensional torus and spatial domains with finite volume and nonincoming boundary conditions. Some L^1 "averaging lemmas" are also given.

1 Introduction

A main feature of spectra of transport operators in nuclear reactor theory relies on the compactness (or weak compactness in L^1) of *some power* of $K(\lambda - T)^{-1}$ where T denotes the advection operator

$$T\varphi = -v\frac{\partial \varphi}{\partial x} - \sigma(x,v)\varphi$$

with suitable boundary conditions and K is the collision operator which describes the interactions of neutrons with the host medium. Indeed, according to Gohberg-Schmulyan's theorem [13], $\sigma(T+K) \cap \{\operatorname{Re}\lambda > s(T)\}$ (the so-called *asymptotic spectrum* of T) consists of at most isolated eigenvalues with finite algebraic multiplicities where $s(T)$ is the spectral bound of T

$$s(T) = \sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(T)\}.$$

On the other hand, the time asymptotic behavior ($t \to \infty$) of the c_0-semigroup $\{V(t);\ t \geq 0\}$ generated by $T + K$, which governs the Cauchy problem

$$\frac{\partial \varphi}{\partial t} + v\frac{\partial \varphi}{\partial x} + \sigma(x,v)\varphi + K\varphi = 0, \;\; \varphi(0) = \varphi_0,$$

depends heavily on the spectrum of $\{V(t);\ t \geq 0\}$ *outside* the disc

$$\left\{\nu; |\nu| \leq e^{s(T)t}\right\},$$

(see [14]). Of course,

$$e^{t\{\sigma(T+K)\cap\{\operatorname{Re}\lambda > s(T)\}\}} \subset \sigma(e^{t(T+K)}) \cap \left\{\nu; |\nu| > e^{s(T)t}\right\}. \tag{1}$$

However, this inclusion is a priori *strict* because of the lack, in general, of a spectral mapping theorem. Thus a *direct* spectral analysis of $e^{t(T+K)}$ is necessary. To this end, we expand $V(t)$ into a Dyson-Philips expansion

$$V(t) = \sum_{0}^{\infty} U_j(t)$$

where

$$U_0(t) = e^{tT}, \;\; U_{j+1}(t) = \int_0^t U_0(s)KU_j(t-s)ds \;\; (j \geq 0).$$

A basic result is that (1) is an *equality* if some remainder term $R_m(t)$ is compact (or weakly compact in L^1) where

$$R_m(t) = \sum_{j=m}^{\infty} U_j(t)$$

(see [14] [17] [18] [19] [11] and [7] Chap 2 for more details). In such a case, $\sigma(e^{t(T+K)}) \cap \left\{\nu; |\nu| > e^{s(T)t}\right\}$ (the so-called asymptotic spectrum of $V(t)$) consists of, at most, isolated eigenvalues with finite algebraic multiplicities. Thus, the asymptotic spectral theory of the transport operator T relies on the *compactness of some power of* $K(\lambda - T)^{-1}$ while the asymptotic spectral theory of the corresponding semigroup relies on the *compactness of some remainder term* $R_m(t)$. These are the two basic compactness problems in neutron transport theory. Of course, there exists a great deal of works on this topic since the fifties already covering all the usual models (see [7] Chap 4 and references therein). In a recent work [9] the author gave *necessary and sufficient* compactness results for tranport equations in L^p spaces $(1 < p < \infty)$ in terms of properties of the velocity measure. This provides us with an *optimal* spectral theory for neutron tranport equations for both periodic boundary conditions and classical nonincoming boundary conditions. The mathematical analysis relies on "*Fourier integral*" type arguments. This approach, of course, does not cover the (physical) L^1 spaces. The present paper deals with the L^1 theory. We obtain nearly optimal theorems by using *new* mathematical tools. Indeed, some relevant operators are shown to be convolution operators with suitable Radon measures. The Fourier analysis of such measures enables us to derive smoothing properties of their *convolution iterates* from which various weak compactness results are obtained. In Section 2 and Section 3, we deal with transport equations with *model* collision operators on the n-dimensional torus. A thorough analysis of the different aspects of (weak) compactness is given with detailled proofs. In Section 4 and Section 5, we treat transport equations on domains Ω with *finite* volume (not necessarilly bounded) and nonincoming boundary conditions; the treatment is quite similar (with some modifications) and the proofs are only sketched. In Section 6 we give much more precise results (similar to that of the L^p theory [9]) in *one dimension* and show that these results are no longer true in n dimensions with $n \geq 3$. In Section 7 we show how the above compactness results provide a complete foundation of the L^1 spectral theory of neutron transport equations for general collision operators. Although they have not a direct connection with the main purpose of this paper, we give in the last section some L^1 "averaging lemmas" which improve or complement some known results.

2 Model stationary equations on the torus

Let Ω be the n-dimensional torus ($n \geq 1$) we identify with $[0, 2\pi]^n$. *We identify $L^1(\Omega)$ with the locally integrable $[2\pi]^n$-periodic functions on R^n.* Similarly, $C(\Omega)$ denotes the continuous $[2\pi]^n$-periodic functions on R^n. Let $d\mu$ be a positive finite Radon measure on R^n with support V. Let $\{U(t);\ t \in R\}$ be the c_0-group of isometries

$$U(t) : \varphi \in L^1(\Omega \times V) \to \varphi(x - tv, v) \in L^1(\Omega \times V)$$

where $\Omega \times V$ is endowed with the product measure $dx \otimes d\mu$. The infinitesimal generator of $\{U(t);\ t \in R\}$ is given by

$$T : \varphi \in D(T) \to -v.\frac{\partial \varphi}{\partial x} \in L^1(\Omega \times V)$$

with

$$D(T)=\left\{\varphi\in L^1(\Omega\times V);\ v.\frac{\partial\varphi}{\partial x}\in L^1(\Omega\times V)\right\}$$

where the directional derivative $v.\frac{\partial\varphi}{\partial x}$ is taken in the sense of periodic distribution. The resolvent of T, for $\lambda>0$, is given by

$$(\lambda-T)^{-1}:\varphi\in L^1(\Omega\times V)\to\int_0^\infty e^{-\lambda t}\varphi(x-tv,v)dt.$$

We are concerned with the smoothing properties of $M(\lambda-T)^{-1}$ where

$$M:\varphi\in L^1(\Omega\times V)\to\widetilde{\varphi}(.):=\int\varphi(.,v)\ d\mu(v)\in L^1(\Omega)\tag{2}$$

is the so called (velocity) *averaging* operator. More precisely, we are looking for necessary and (or) sufficient conditions on the measure $d\mu$ such that *some power* of $M(\lambda-T)^{-1}$ is weakly compact or compact. We start with the following result which was first pointed out in ([2] Prop 3 and example 1) for the whole space.

Proposition 1 *(i) The operator*

$$M(\lambda-T)^{-1}:L^1(\Omega\times V)\to L^1(\Omega)$$

is not weakly compact.

(ii) If the hyperplanes through the origin have zero $d\mu$-measure then $M(\lambda-T)^{-1}$ maps weakly compact sets into compact sets.

Proof:

The proof is the same as that given in [2]. However, for the reader's convenience, we resume it here.

(i) Let $\{f_j\}_j\subset L^1(\Omega\times V)$ be a normalized sequence converging in the weak star topology of measures to the Dirac mass $\delta_{(0,\overline{v})}=\delta_{x=0}\otimes\delta_{v=\overline{v}}$ where

$\overline{v} \in V$. Then, for a $\psi \in C(\Omega)$,

$$\begin{aligned}
\langle M(\lambda - T)^{-1} f_j, \psi \rangle &= \int_\Omega \psi(x)dx \int d\mu(v) \int_0^\infty e^{-\lambda t} f_j(x - tv, v)dt \\
&= \int_0^\infty e^{-\lambda t} dt \int d\mu(v) \int_{\Omega - tv} \psi(y + tv) f_j(y, v)dy \\
&= \int_0^\infty e^{-\lambda t} dt \int d\mu(v) \int_\Omega \psi(y + tv) f_j(y, v)dy \\
&= \int_0^\infty e^{-\lambda t} dt \int_{\Omega \times V} \psi(y + tv) f_j(y, v)dy d\mu(v) \\
&= \int_{\Omega \times V} \left[\int_0^\infty e^{-\lambda t} \psi(y + tv)dt \right] f_j(y, v)dy d\mu(v)
\end{aligned}$$

and

$$\langle M(\lambda - T)^{-1} f_j, \psi \rangle \to \int_0^\infty e^{-\lambda t} \psi(t\overline{v})dt \quad \text{as } j \to \infty$$

i.e. $M(\lambda - T)^{-1} f_j$ converges to the Radon measure

$$\psi \in C(\Omega) \to \int_0^\infty e^{-\lambda t} \psi(t\overline{v})dt$$

supported on the *line* $R\overline{v}$ and consequently $M(\lambda - T)^{-1} f$ is *not* weakly compact if $n > 1$. If $n = 1$ and if $0 \in V$ then the choice $\overline{v} = 0$ shows that $M(\lambda - T)^{-1} f_j$ converges to the Dirac measure $\frac{1}{\lambda}\delta_{x=0}$. Of course, if $n = 1$ and if $0 \notin V$ it is easy to see that $M(\lambda - T)^{-1}$ is a *compact* operator.

(ii) Let $\Xi \subset L^1(\Omega \times V)$ be relatively weakly compact. We have to prove that if $g = M(\lambda - T)^{-1} f,\ f \in \Xi$, then

$$\int_\Omega |g(x + h) - g(x)|\, dx \to 0 \ \text{ uniformly in } \ f \in \Xi \tag{3}$$

as $h \to 0$. We write $g = g_1 + g_2$ where

$$g_1 = M(\lambda - T)^{-1}(f\chi_{\{f > \alpha\}}) \text{ and } g_2 = M(\lambda - T)^{-1}(f\chi_{\{f < \alpha\}}).$$

We note that

$$\int_\Omega |g_1(x + h) - g_1(x)|\, dx \leq 2\, \|g_1\| \leq \frac{2\, \|M\|}{\lambda} \int_{\{f > \alpha\}} |f(x, v)|\, dx d\mu(v)$$

and

$$dxd\mu\{f>\alpha\} \leq \frac{\|f\|}{\alpha} \leq \frac{c}{\alpha} \to 0$$

so that, by the *equi-integrability* of Ξ,

$$\int_{\{f>\alpha\}} |f(x,v)|\, dxd\mu(v) \to 0 \ \text{ uniformly in } \ f \in \Xi$$

as $\alpha \to \infty$. Thus, for $\varepsilon > 0$,

$$\int_{\Omega} |g_1(x+h) - g_1(x)|\, dx \leq \varepsilon \text{ uniformly in } \ f \in \Xi$$

for α large enough. We fix this α. Then $\left\{f\chi_{\{f<\alpha\}};\ f \in \Xi\right\}$ is a bounded subset of $L^2(\Omega \times V)$ and consequently $\{g_2;\ f \in \Xi\}$ is relatively compact in $L^2(\Omega)$ (see [9] Thm 9) and consequently relatively compact in $L^1(\Omega)$ so that

$$\int_{\Omega} |g_2(x+h) - g_2(x)|\, dx \to 0 \ \text{ uniformly in } \ f \in \Xi$$

as $h \to 0$. This proves (3). $\square$

Before giving our compactness results we derive a *necessary* condition.

Proposition 2 *We assume that $d\mu$ is invariant under the symmetry about the origin $v \to -v$. If some power of $M(\lambda - T)^{-1}$ is weakly compact then the hyperplanes through the origin have zero $d\mu$-measure.*

Proof:

Since the square of a weakly compact operator in L^1 is compact [1], we may assume that some power of $M(\lambda - T)^{-1}$ is compact. Then some power of $M(\lambda - T)^{-1}M$ is also compact. On the other hand, since $M(\lambda - T)^{-1}M$ maps also $L^p(\Omega \times V)$ into $L^p(\Omega)$ for all $p \in [1, \infty]$ then, by interpolation, some power of

$$M(\lambda - T)^{-1}M : L^2(\Omega \times V) \to L^2(\Omega)$$

is compact too. We may assume, without loss of generality that

$$\left[M(\lambda - T)^{-1}M\right]^{2^m} : L^2(\Omega \times V) \to L^2(\Omega)$$

is compact for some integer m. On the other hand, $M(\lambda - T)^{-1}M$ is *selfadjoint* for λ *real.* Indeed,

$$\begin{aligned}
(M(\lambda - T)^{-1}M\varphi, \psi) &= ((\lambda - T)^{-1}M\varphi, M\psi) \\
&= \int_\Omega \int_V dx d\mu(v) \int_0^\infty e^{-\lambda t}(M\varphi)(x - tv)dt\overline{(M\psi)(x)} \\
&= \int_V d\mu(v) \int_0^\infty e^{-\lambda t} \int_\Omega dx(M\varphi)(x - tv)dt\overline{(M\psi)(x)} \\
&= \int_V d\mu(v) \int_0^\infty e^{-\lambda t} \int_\Omega dy(M\varphi)(y)dt\overline{(M\psi)(y + tv)} \\
&= \int_0^\infty e^{-\lambda t} \int_\Omega dy(M\varphi)(y)dt \int_V d\mu(v)\overline{(M\psi)(y + tv)} \\
&= \int_0^\infty e^{-\lambda t} \int_\Omega dy(M\varphi)(y)dt \int_V d\mu(v)\overline{(M\psi)(y - tv)} \\
&= \int_\Omega \int_V dy d\mu(v)(M\varphi)(y) \int_0^\infty e^{-\lambda t}\overline{(M\psi)(y - tv)}dt \\
&= (M\varphi, (\lambda - T)^{-1}M\psi) = (\varphi, M(\lambda - T)^{-1}M\psi).
\end{aligned}$$

Hence the compactness of $[M(\lambda - T)^{-1}M]^{2^m}$ implies the compactness of $[M(\lambda - T)^{-1}M]^{2^{m-1}}$ by the fact that the square of a *selfadjoint* operator O is compact if and only if O is. It follows, by induction, that $M(\lambda - T)^{-1}M$ is compact. We use now Vladimirov's argument [15] as in [6] to prove that $(\lambda - T)^{-1}M$ is compact. It follows that $M(\lambda - T^*)^{-1}$ is compact and this implies that the hyperplanes through the origin have *zero* $d\mu$-measure ([7] Remark 3.1, p. 35). $\square$

From now on we assume that

$$\text{The hyperplanes through the origin have zero } d\mu\text{-measure.} \tag{4}$$

If we except the dimension *one* (see Section 6), Assumption (4) alone does not seem to be sufficient to derive *compactness* results (see however [8] for Dunford-Pettis results). However, some slightly stronger condition will be. To this end, we recall the following:

Lemma 1 *([7] lemma 3.1, p. 32) All the hyperplanes through the origin have zero $d\mu$-measure if and only if* $\sup_{e \in S^{n-1}} d\mu \{v;\ |v.e| \leq \varepsilon\} \to 0$ *as* $\varepsilon \to 0$.

A key point in our subsequent analysis is that $M(\lambda - T)^{-1}M$ is a *convolution operator with a suitable Radon measure $d\beta$* whose Fourier properties turn out to play a crucial role. The fact to interpret various operators (related to tranport equations) as convolution with suitable measures was introduced by the author in ([7] Chap 4) but was not fully exploited.

Lemma 2 *There exists a Radon measure $d\beta$ on R^n such that*

$$M(\lambda - T)^{-1}M\varphi = \int_{R^n} (M\varphi)(x-y)d\beta(y) = d\beta * M\varphi.$$

Moreover, the Fourier transform of $d\beta$ is given by

$$\widehat{d\beta}(\zeta) = \int_{R^n} e^{-i\zeta.y} d\beta(y) = \int \frac{d\mu(v)}{\lambda + i\zeta.v} \ \ (\zeta \in R^n). \tag{5}$$

Proof:

We point out that the above convolution *does not take place on the torus* but on R^n. Moreover,

$$d\beta * M\varphi \in L^1(\Omega).$$

We note that

$$\begin{aligned} M(\lambda - T)^{-1}M\varphi &= \int_0^\infty e^{-\lambda t}dt \int_{R^n} (M\varphi)(x - tv)\ d\mu(v) \\ &= \int_0^\infty e^{-\lambda t}dt \int_{R^n} (M\varphi)(x - z)\ d\mu_t(z) \end{aligned}$$

where $d\mu_t$ is the image of $d\mu$ under the dilation $v \to tv$. Hence

$$M(\lambda - T)^{-1}M\varphi = \int (M\varphi)(x-z)\ d\beta(z) = d\beta * M\varphi \tag{6}$$

where

$$d\beta = \int_0^\infty e^{-\lambda t} d\mu_t dt$$

denotes the measure

$$\psi \in C(\Omega) \to \int_0^\infty e^{-\lambda t}\langle d\mu_t, \psi\rangle dt.$$

Morevoer, the k^{th} Fourier coefficient of the $L^1(\Omega)$-function $M(\lambda - T)^{-1}M\varphi$ is equal to

$$\begin{aligned} &\int_\Omega e^{-ik.x}dx \int_0^\infty e^{-\lambda t}dt \int_{R^n} (M\varphi)(x - tv)\ d\mu(v) \\ = &\int_0^\infty e^{-\lambda t}dt \int_{R^n} e^{-itk.v}\widehat{M\varphi}_k\ d\mu(v) = (\int_{R^n} \frac{d\mu(v)}{\lambda + ik.v})\widehat{M\varphi}_k \\ = &\ \widehat{d\beta}(k)\widehat{M\varphi}_k \end{aligned}$$

where $\widehat{M\varphi}_k$ is the k^{th} Fourier coefficient of the $L^1(\Omega)$-function $M\varphi$ and $\widehat{d\beta}(k)$ is the *continuous* Fourier transform of $d\beta$ on R^n evaluated at $k \in Z^n$. $\square$

Remark 1 *Assumption* (4) *that hyperplanes have zero $d\mu$-measure implies $\int_{R^n} \frac{d\mu(v)}{\lambda+i\zeta.v} \to 0$ as $|\zeta| \to \infty$ (see, for instance,* [7] *Chap 3), i.e. $\widehat{d\beta}(\zeta) \to 0$ as $|\zeta| \to \infty$. In particular*

$$\widehat{d\beta}(k) \to 0 \text{ as } |k| \to \infty \ \ (k \in Z^n). \tag{7}$$

We are going to show that a slightly stronger assumption than (7) *is the key of the problem.*

Theorem 1 *We assume there exists $s \geq 1$ such that*

$$\sum_{k \in Z^n} \left| \widehat{d\beta}(k) \right|^s < \infty. \tag{8}$$

Let m be the least integer such that (8) *is satisfied with $s = 2m$. Then $[M(\lambda - T)^{-1}]^{m+1}$ is weakly compact and $[M(\lambda - T)^{-1}]^{m+2}$ is compact.*

Proof:
According to Lemma 2

$$\begin{aligned} \left[M(\lambda - T)^{-1}M\right]^2 \varphi &= d\beta * [M(d\beta * M\varphi)] \\ &= \|d\mu\| \, d\beta * (d\beta * M\varphi) \\ &= \|d\mu\| \, (d\beta * d\beta) * M\varphi. \end{aligned}$$

We show by induction that

$$\left[M(\lambda - T)^{-1}M\right]^m \varphi = \|d\mu\|^{m-1} \, d\nu * M\varphi$$

where $d\nu = d\beta * d\beta * \cdots * d\beta$ (m times). Hence the k^{th} Fourier coefficient of $[M(\lambda - T)^{-1}M]^m \varphi$ is equal to

$$\|d\mu\|^{m-1} \, \widehat{d\nu}(k) \widehat{M\varphi}_k = \|d\mu\|^{m-1} \left[\widehat{d\beta}(k)\right]^m \widehat{M\varphi}_k.$$

On the other hand, according to (8), $\left\{ \left[\widehat{d\beta}(k)\right]^m \right\}_k \in l^2(Z^n)$ and consequently $\left\{ \left[\widehat{d\beta}(k)\right]^m \widehat{M\varphi}_k \right\}_k \in l^2(Z^n)$ since $\{\widehat{M\varphi}_k\}_k \in c_0(Z^n)$. Then Parseval identity yields

$$\left[M(\lambda - T)^{-1}M\right]^m \varphi \in L^2(\Omega).$$

This shows that $[M(\lambda - T)^{-1}M]^m$ maps continuously $L^1(\Omega \times V)$ into $L^2(\Omega)$ and consequently

$$\left[M(\lambda - T)^{-1}M\right]^m : L^1(\Omega \times V) \to L^1(\Omega)$$

is *weakly* compact since the injection of $L^2(\Omega)$ in $L^1(\Omega)$ is *weakly* compact by the Dunford-Pettis criterion of weak compactness. We note that $M^2 = \|d\mu\| M$ and consequently $[M(\lambda - T)^{-1}]^{m+1}$ is weakly compact in $L^1(\Omega \times V)$, i.e. maps bounded sets into weakly compact ones and consequently $[M(\lambda - T)^{-1}]^{m+2}$ is compact since, by Prop 1, $M(\lambda - T)^{-1}$ maps weakly compact sets into compact sets. $\square$

Remark 2 *Is* (8) *true for all* $d\mu$ *satisfying* (4) *? If not, is it possible to characterize those measures satisfying* (8) *? A sufficient condition is provided by the following:*

Proposition 3 *We suppose there exist* $0 < \gamma < 1$ *and* $\delta \geq 1$ *such that*

$$\sum_{k \in Z^n} \left[\sup_{e \in S^{n-1}} d\mu \left\{ |v.e| \leq \frac{1}{|k|^\gamma} \right\} \right]^\delta < \infty. \tag{9}$$

Then (8) *is satisfied for even integer* $s = 2m > \max\left\{\delta, \frac{n}{1-\gamma}\right\}$. *In particular, if there exist* $\alpha > 0$ *and* $c > 0$ *such that*

$$\sup_{e \in S^{n-1}} d\mu \{v;\ |v.e| \leq \varepsilon\} \leq c\varepsilon^\alpha \tag{10}$$

then (9) *is satisfied.*

Proof:
We note that

$$\left|\widehat{d\beta}(k)\right| \leq \int \frac{d\mu(v)}{|\lambda + ik.v|} = \int \frac{d\mu(v)}{\sqrt{\lambda^2 + |k|^2 |e.v|^2}}$$

where $e = \frac{k}{|k|} \in S^{n-1}$. Thus

$$\begin{aligned}
\left|\widehat{d\beta}(k)\right| &\leq \int_{|e.v| \leq \varepsilon} \frac{d\mu(v)}{\sqrt{\lambda^2 + |k|^2 |e.v|^2}} + \int_{|e.v| > \varepsilon} \frac{d\mu(v)}{\sqrt{\lambda^2 + |k|^2 |e.v|^2}} \\
&\leq \frac{1}{\lambda} d\mu \{|v.e| \leq \varepsilon\} + \frac{\|d\mu\|}{\sqrt{\lambda^2 + |k|^2 \varepsilon^2}}.
\end{aligned}$$

Choose $\varepsilon = \frac{1}{|k|^{\gamma}}$. Then, for $k \neq 0$,

$$\frac{\|d\mu\|}{\sqrt{\lambda^2 + |k|^2 \varepsilon^2}} \leq \frac{\|d\mu\|}{|k| \varepsilon} = \frac{\|d\mu\|}{|k|^{1-\gamma}}$$

so $\left\{ \frac{\|d\mu\|}{\sqrt{\lambda^2+|k|^2\varepsilon^2}} \right\}_k \in l^{2m}(Z^n)$ if $2(1-\gamma)m > n$, i.e. for all $m > \frac{n}{2(1-\gamma)}$. Moreover, according to (9),

$$\left\{ \sup_{e \in S^{n-1}} d\mu \left\{ |v.e| \leq \frac{1}{|k|^{\gamma}} \right\} \right\}_k \in l^{2m}(Z^n) \text{ if } 2m \geq \delta$$

whence $\left\{ \widehat{d\beta}(k) \right\}_k \in l^{2m}(Z^n)$ if $2m > \max \left\{ \delta, \frac{n}{1-\gamma} \right\}$. $\square$

Remark 3 *Condition* (10) *in Prop 3 is obviously satisfied by Lebesgue measures on bounded open sets or on spheres.*

3 On model evolution equations on the torus

We deal now with the c_0-group $\{V(t); t \in R\}$ generated by $T+M$ where M is the velocity averaging operator (2). We recall that this perturbed group is given by a Dyson-Philips expansion

$$V(t) = \sum_{j=0}^{\infty} U_j(t) \tag{11}$$

where

$$U_0(t) = U(t) \text{ and } U_j(t) = \int_0^t U(t-s) M U_{j-1}(s) ds \quad (j \geq 1).$$

Let $R_m(t) = \sum_{j=m}^{\infty} U_j(t)$ $(m \geq 1)$ be the remainder terms of the Dyson-Philips expansion (11). We are concerned in this section with conditions on the velocity measure $d\mu$ under which some remainder term $R_m(t)$ is *weakly compact*. We observe that $U_j = [UM]^j * U \quad (j \geq 1)$ where $*$ is the convolution operator which associates to strongly continuous (operator valued) mappings

$$f, g : [0, \infty[\to L(L^1(\Omega \times V))$$

the strongly continuous mapping

$$f * g : t \in [0, \infty[\rightarrow \int_0^t f(t-s)g(s)ds \in L(L^1(\Omega \times V))$$

and $[UM]^j = (UM) * \cdots * (UM)$ (j times). We note that: $f, g \rightarrow f * g$ is associative. We recall (see [7] Chap 2, Thm 2.6, p. 16) that $R_m(t)$ is weakly compact for all $t \geq 0$ *if and only if* $U_m(t)$ is. According to the convex compactness property of the strong operator topology ([12] or [8]), the weak compactness of $[UM]^m(t)$ for all $t \geq 0$ implies the weak compactness of

$$U_m(t) = \int_0^t [UM]^m(s)U(t-s)ds.$$

Thus, we may deal with

$$[UM]^m = [UM] * [UM] \cdots * [UM] \ \ (m \text{ times}).$$

On the other hand, since $M^2 = \|d\mu\| M$, one sees that

$$[UM]^m(t) = \frac{1}{\|d\mu\|^{m-2}} U * [MUM] \cdots * [MUM]$$

where the term $[MUM]$ appears $m-1$ times. By appealing againg to the convex compactness property of the strong operator topology, we may deal with the weak compactness of $[MUM]^{m-1}$. The basic strategy in this section relies on the fact that $[MUM]^{m-1}$ is a convolution operator with a suitable Radon measure whose Fourier properties will play a key role.

Lemma 3 *Let $m \in N$ ($m \geq 1$). There exists a Radon measure $d\beta^m$ on R^n such that*

$$[MUM]^m \varphi = d\beta^m * M\varphi. \tag{12}$$

Proof:

Arguing as in the proof of Lemma 2,

$$MU(t)M\varphi = \int M\varphi(x-tv)d\mu(v) = \int M\varphi(x-y)d\mu_t(y) = d\mu_t * M\varphi$$

where $d\mu_t$ is the image of $d\mu$ under the dilation $v \rightarrow tv$. Note again that the convolution above takes place on R^n. Observe that the mapping $t > 0 \rightarrow d\mu_t \in M(\Omega)$ is *strongly* continuous, i.e.

$$t > 0 \rightarrow \langle d\mu_t, \varphi \rangle = \int \varphi(x-tv)d\mu(v)$$

is continuous. We have

$$\begin{aligned}
[MUM]^2(t)\varphi &= \int_0^t MU(t-s)MMU(s)M\varphi ds \\
&= \int_0^t d\mu_{t-s} * M(MU(s)M\varphi)ds \\
&= \|d\mu\| \int_0^t d\mu_{t-s} * (d\mu_s * M\varphi)ds \\
&= \|d\mu\| \int_0^t (d\mu_{t-s} * d\mu_s) * M\varphi ds \\
&= \|d\mu\| \left[\int_0^t (d\mu_{t-s} * d\mu_s)ds\right] * M\varphi. \\
&= \|d\mu\| \, d\beta^2(t) * M\varphi
\end{aligned}$$

where the integral

$$d\beta^2(t) = \int_0^t (d\mu_{t-s} * d\mu_s)ds$$

is taken in the strong sense, i.e.

$$\langle d\beta^2(t), \varphi\rangle = \int_0^t \langle d\mu_{t-s} * d\mu_s, \varphi\rangle ds.$$

One sees, by induction, that

$$[MUM]^m(t)\varphi = \|d\mu\|^{m-1} d\beta^m(t) * M\varphi \tag{13}$$

where $d\beta^m(t)$ is defined inductively by

$$d\beta^m(t) = \int_0^t (d\mu_{t-s} * d\beta^{m-1}(s))ds \quad (m > 2)$$

which ends the proof. $\square$

Before stating the main result of this section we recall ([9] Lemma 2) that the *affine* (i.e. translated) hyperplanes have zero $d\mu$-measure if and only if

$$\sup_{e\in S^{n-1}} d\mu \otimes d\mu \{(v,v') \in V \times V;\ |(v-v').e| < \varepsilon\} \to 0 \ as\ \varepsilon \to 0. \tag{14}$$

We are going to show that a slightly stronger condition than (14) is the *key* of the problem.

Theorem 2 *We assume there exist* $0 < \tau < 1$ *and* $\delta \geq 1$ *such that*

$$\sum_{k \in Z_*^n} \left[\sup_{e \in S^{n-1}} d\mu \otimes d\mu \left\{ (v, v') \in V \times V;\ |(v - v').e| < \frac{1}{|k|^\tau} \right\} \right]^\delta < \infty \quad (15)$$

where $Z_*^n = Z^n - \{0\}$. *Let* m *be the least even integer such that*

$$m > \max \left\{ \delta, \frac{n}{(1-\tau)} \right\}.$$

Then the remainder terms $R_j(t)$ *are weakly compact for all* $t \geq 0$ *and* $j \geq m+1$. *In particular,* (15) *is satisfied if there exist* $c > 0$ *and* $\alpha > 0$ *such that*

$$\sup_{e \in S^{n-1}} d\mu \otimes d\mu \{ (v, v') \in V \times V;\ |(v - v').e| < \varepsilon \} \leq c\varepsilon^\alpha. \quad (16)$$

Proof:

It suffices to prove there exists an integer $j > 1$ such that $[MUM]^{j-1}(t)$ is weakly compact for all $t \geq 0$. Set $j - 1 = m$. We look for an *even* integer m, i.e. $m = 2p$. In such a case, $[MUM]^m = \left[[MUM]^2 \right]^p$ where

$$[MUM]^2 : \varphi \in L^1(\Omega \times V) \to \|d\mu\| \, d\beta^2(t) * M\varphi \in L^1(\Omega)$$

and

$$d\beta^2(t) = \int_0^t d\mu_s * d\mu_{t-s} ds.$$

A simple calculation shows that the *continuous* Fourier transform of $d\beta^2(t)$ evaluated at $k \in Z^n$ is equal to

$$\widehat{d\beta(t)^2}(k) = \int_0^t \widehat{d\mu_s}(k) \widehat{d\mu_{t-s}}(k) ds$$

where $\widehat{d\mu_s}$ is the *continuous* Fourier transform of $d\mu_s$. Thus

$$\begin{aligned}
\widehat{d\beta(t)^2}(k) &= \int_0^t \left[\int e^{-iv.k} d\mu_s(v) \right] \left[\int e^{-iv'.k} d\mu_{t-s}(v') \right] ds \\
&= \int_0^t \left[\int e^{-isv.k} d\mu(v) \right] \left[\int e^{-i(t-s)v'.k} d\mu(v') \right] ds \\
&= \int \int \left[\int_0^t e^{-isv.k} e^{-i(t-s)v'.k} ds \right] d\mu(v) d\mu(v') \\
&= \int \int \frac{e^{-itv'.k} - e^{-itv.k}}{i(v - v').k} d\mu(v) d\mu(v').
\end{aligned}$$

Introducing the polar coordinates $k = |k|\, e$, $e \in S^{n-1}$, we decompose the last integral as

$$\int\int_{|(v-v').e|\leq\varepsilon} \frac{e^{-itv'.k} - e^{-itv.k}}{i(v-v').k} + \int\int_{|(v-v').e|>\varepsilon} \frac{e^{-itv'.k} - e^{-itv.k}}{i(v-v').k}$$

where $\varepsilon > 0$ is arbitrary. Thus

$$\left|\widehat{d\beta(t)^2}(k)\right| \leq c_t \int\int_{|(v-v').e|\leq\varepsilon} d\mu(v)d\mu(v') + \frac{2}{|k|\,\varepsilon}\int\int d\mu(v)d\mu(v')$$

where

$$c_t = t \sup_{p\neq q}\left|\frac{e^{ip}-e^{iq}}{p-q}\right|. \tag{17}$$

Let $0 < \tau < 1$ and $\varepsilon = |k|^{-\tau}$. Hence $\left|\widehat{d\beta(t)^2}(k)\right|$ is majorized by

$$c_t \sup_{e\in S^{n-1}} d\mu \otimes d\mu \left\{(v,v') \in V \times V;\ |(v-v').e| < \frac{1}{|k|^{\tau}}\right\} \tag{18}$$

$$+\frac{2\,\|d\mu\|^2}{|k|^{1-\tau}} \quad = \quad c_t a_k + b_k$$

where $b_k = \frac{2\|d\mu\|^2}{|k|^{1-\tau}}$. Note that $\{a_k\}_k$ and $\{b_k\}_k$ *do not* depend on t. Clearly, $\{b_k\}_k \in l^q(Z^n)$ for all $q > \frac{n}{(1-\tau)}$. On the other hand, according to (15), $\{a_k\}_k \in l^\delta(Z^n)$ and consequently

$$\{a_k + b_k\} \in l^r(Z^n)\ \ \forall r > \max\left\{\delta, \frac{n}{(1-\tau)}\right\}. \tag{19}$$

According to (12)

$$[MUM]^4 : \varphi \in L^1(\Omega\times V) \to \|d\mu\|^3\, d\beta^4(t) * M\varphi \in L^1(\Omega)$$

where

$$d\beta^4(t) = \int_0^t d\beta^2(t-s) * d\beta^2(s)ds$$

whence

$$\widehat{d\beta^4(t)}(k) = \int_0^t d\widehat{\beta(t-s)^2}(k)\widehat{d\beta(s)^2}(k)ds$$

and

$$\begin{aligned}\left|\widehat{d\beta^4(t)}(k)\right| &\leq \int_0^t \left|\widehat{d\beta(t-s)^2}(k)\right| \left|\widehat{d\beta(s)^2}(k)\right| ds \\ &\leq \int_0^t (c_{t-s}a_k + b_k)(c_s a_k + b_k) ds \\ &\leq t c_t'^2 (a_k + b_k)^2\end{aligned}$$

where $c_t'^2 := \max\{1, c_t\}$. It follows, by induction, that

$$\left|\widehat{d\beta^{2p}(t)}(k)\right| \leq \widehat{c}(p,t)(a_k + b_k)^p$$

where $\widehat{c}(p,t)$ is a constant (in k) depending *only* on t and p. According to (19)

$$\{(a_k + b_k)^p\} \in l^{\frac{r}{p}}(Z^n) \;\; \forall r > \max\left\{\delta, \frac{n}{(1-\tau)}\right\}.$$

By choosing an integer $p \geq \frac{r}{2}$, we have $\frac{r}{p} \leq 2$ and therefore

$$\{(a_k + b_k)^p\} \in l^2(Z^n).$$

Hence, for $m = 2p > \max\left\{\delta, \frac{n}{(1-\tau)}\right\}$, $\left\{\widehat{d\beta^m(t)}(k)\right\} \in l^2(Z^n)$. On the other hand

$$[MUM]^m \varphi = d\beta^m * M\varphi$$

shows that the k^{th} Fourier coefficient $\widehat{d\beta^m(t)}(k)\widehat{M\varphi}_k$ of the $L^1(\Omega)$-function $[MUM]^m \varphi$ is majorized by

$$\left|\widehat{d\beta^m(t)}(k)\right| \|M\varphi\|_{L^1(\Omega)} \in l^2(Z^n)$$

so that, by Parseval identity, $[MUM]^m \varphi \in L^2(\Omega)$. Hence, for $m = 2p > \max\left\{\delta, \frac{n}{(1-\tau)}\right\}$, $[MUM]^m$ maps continuously $L^1(\Omega \times V)$ into $L^2(\Omega)$ so that

$$[MUM]^m : L^1(\Omega \times V) \to L^1(\Omega)$$

is *weakly* compact. Finally, $[MUM]^{j-1}$ is *weakly* compact for $j - 1 > \max\left\{\delta, \frac{n}{(1-\tau)}\right\}$ and so is $R_j(t)$. On the other hand, since

$$R_{i+1}(t) = \int_0^t U(t-s) M R_i(s) ds \;\; (i \geq 1)$$

([7] Lemma 2.2, p.15) then, by the convex compactness property of the strong operator topology ([12] or [8]), it follows that $R_i(t)$ is weakly compact for all $i \geq j$.□

Remark 4 *We point out that the weak compactness of some remainder term $R_m(t)$ for all $t \geq 0$ implies the compactness of $R_{m+2}(t)$ (see* [8]*). Condition* (16) *in Thm 2 is obviously satisfied by Lebesgue measures on bounded open sets or on spheres.*

4 Model stationary equations with nonincoming boundary conditions

Let $\Omega \subset R^n$ be an open set with *finite Lebesgue measure* (not necessarily bounded) and $d\mu$ be a finite and positive Radon measure on R^n with support V. We denote by $\{U(t); t \geq 0\}$ the classical advection c_0-semigroup with nonincoming boundary conditions

$$U(t) : \varphi \in L^1(\Omega \times V) \to \varphi(x - tv, v)\chi(t < \tau(x, v)) \in L^1(\Omega \times V)$$

where $\tau(x, v) = \inf \{s > 0,\ x - sv \notin \Omega\}$. Let T be its generator. We *do not* need its description. We note however that if $\partial\Omega$ is "smooth" then

$$T\varphi = -v.\frac{\partial \varphi}{\partial x}\ ;\ \varphi \in D(T)$$

where

$$D(T) = \left\{\varphi \in L^1(\Omega \times V);\ v.\frac{\partial \varphi}{\partial x},\ \varphi_{|\Gamma_-} = 0\right\}$$

$$\Gamma_- := \{(x, v) \in \partial\Omega \times V;\ v.n(x) < 0\}$$

and $n(x)$ is the unit outward normal at $x \in \partial\Omega$ (see, for instance, [16]). Let

$$(\lambda - T)^{-1} : \varphi \in L^1(\Omega \times V) \to \int_0^{\tau(x,v)} e^{-\lambda t}\varphi(x - tv, v)dt \ \ (\lambda > 0)$$

be the resolvent of T and let

$$M : \varphi \in L^1(\Omega \times V) \to \widetilde{\varphi}(.) = \int \varphi(., v)\ d\mu(v) \in L^1(\Omega) \tag{20}$$

be the (velocity) averaging operator. As in Section 2, we are concerned with the weak compactness of *some power* of $M(\lambda - T)^{-1}$ and, similarly, we deal first with the powers of $M(\lambda - T)^{-1}M$. The arguments are quite similar so we do not enter into all the details. We start with the following observation:

Proposition 4 *We assume that $d\mu$ is invariant under the symmetry about the origin $v \to -v$. If some power of $M(\lambda - T)^{-1}$ is weakly compact then the hyperplanes through the origin have zero $d\mu$-measure.*

Proof:

We proceed exactly as in the proof of Prop 2. The main point is to show that $M(\lambda - T)^{-1}M$ is *selfadjoint for* λ real. To this end, we note that

$$\begin{aligned}(M(\lambda - T)^{-1}M\varphi, \psi) &= ((\lambda - T)^{-1}M\varphi, M\psi) \\ &= (\varphi, M(\lambda - T^*)^{-1}M\psi)\end{aligned}$$

where T^* is the *adjoint* of T and

$$(\lambda - T^*)^{-1}\varphi = \int_0^{\tau(x,-v)} e^{-\lambda t}\varphi(x + tv, v)dt.$$

On the other hand

$$\begin{aligned}M(\lambda - T^*)^{-1}M\psi &= \int d\mu(v) \int_0^{\tau(x,-v)} e^{-\lambda t}M\varphi(x + tv)dt \\ &= \int d\mu(v) \int_0^{\tau(x,v)} e^{-\lambda t}M\varphi(x - tv)dt \\ &= M(\lambda - T)^{-1}M\psi\end{aligned}$$

because $d\mu$ is unvariant by the symmetry $v \to -v$. $\square$

Thus it is *natural* to assume (4) for the sequel. We note that

$$U(t)\varphi \leq RU_\infty(t)E\varphi; \quad \varphi \in L^1_+(\Omega \times V) \tag{21}$$

where

$$U_\infty(t)\varphi = \varphi(x - tv, v); \quad \varphi \in L^1(R^n \times V)$$

is the advection c_0-semigroup in the *whole space*,

$$E : L^1(\Omega \times V) \to L^1(R^n \times V)$$

is the trivial *extension* (by zero) to $R^n \times V$ and

$$R : L^1(R^n \times V) \to L^1(\Omega \times V)$$

is the *restriction* operator. It follows that

$$(\lambda - T)^{-1}\varphi \leq R(\lambda - T_\infty)^{-1}E\varphi; \;\; \varphi \in L^1_+(\Omega \times V)$$

where

$$(\lambda - T_\infty)^{-1} : \varphi \in L^1(R^n \times V) \to \int_0^\infty e^{-\lambda t}\varphi(x - tv, v)dt \in L^1(R^n \times V).$$

Since E and R commute with the averaging operator M, it follows that

$$M(\lambda - T)^{-1}M\varphi \leq RM(\lambda - T_\infty)^{-1}ME\varphi.$$

It is easy to see, by induction, that

$$\left[M(\lambda - T)^{-1}M\right]^m \varphi \leq R\left[M(\lambda - T_\infty)^{-1}M\right]^m E\varphi; \quad \varphi \in L^1_+(\Omega \times V). \quad (22)$$

Hence, by a domination argument, it suffices to prove that

$$R\left[M(\lambda - T_\infty)^{-1}M\right]^m : L^1(R^n \times V) \to L^1(\Omega)$$

is weakly compact. To this end, it suffices to show that $[M(\lambda - T_\infty)^{-1}M]^m$ maps continuously $L^1(R^n \times V)$ into $L^2(R^n)$. Indeed, in such a case,

$$R\left[M(\lambda - T_\infty)^{-1}M\right]^m : L^1(R^n \times V) \to L^2(\Omega)$$

is *continuous* and

$$R\left[M(\lambda - T_\infty)^{-1}M\right]^m : L^1(R^n \times V) \to L^1(\Omega)$$

is *weakly compact* because the injection of $L^2(\Omega)$ into $L^1(\Omega)$ is weakly compact since *the Lebesgue measure of* Ω *is finite.* On the other hand, for $\varphi \in L^1(R^n \times V)$

$$\begin{aligned} M(\lambda - T)^{-1}M\varphi &= \int_{R^n} d\mu(v) \int_0^\infty e^{-\lambda t}(M\varphi)(x - tv)dt \\ &= \int_0^\infty e^{-\lambda t}dt \int_{R^n} (M\varphi)(x - tv)d\mu(v) \\ &= \int_0^\infty e^{-\lambda t}dt \int_{R^n} (M\varphi)(x - z)\; d\mu_t(z) \end{aligned}$$

where $d\mu_t$ is the image of $d\mu$ under the dilation $v \to tv$. Hence

$$M(\lambda - T)^{-1} M\varphi = \int (M\varphi)(x - z)\ d\beta(z) = d\beta * M\varphi \tag{23}$$

where

$$d\beta = \int_0^\infty e^{-\lambda t} d\mu_t dt.$$

Moreover, the Fourier transform of the L^1 function $M(\lambda - T)^{-1} M\varphi$ is equal to

$$\int_0^\infty e^{-\lambda t} dt \int_{R^n} e^{-it\zeta.v} \widehat{M\varphi}(\zeta)\ d\mu(v) = \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}.\widehat{M\varphi}(\zeta).$$

Hence

$$\widehat{d\beta}(\zeta) = \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}.$$

It follows that

$$\left[M(\lambda - T)^{-1} M\right]^m \varphi = \|d\mu\|^{m-1}\, d\nu * M\varphi$$

where $d\nu = d\beta * \cdots * d\beta$ (m times) and

$$\widehat{d\nu}(\zeta) = \left[\int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}\right]^m.$$

Before stating the main result of this section, we recall again that Assumption (4) that hyperplanes have zero $d\mu$-measure implies

$$\int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v} \to 0 \text{ as } |\zeta| \to \infty.$$

We are going to show that a slightly stronger condition is the *key* of the problem.

Theorem 3 *We assume that Ω has a finite Lebesgue measure and there exists an integer m such that*

$$\int d\zeta \left| \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v} \right|^{2m} < \infty. \tag{24}$$

Then $[M(\lambda - T)^{-1} M]^m$ is weakly compact in $L^1(\Omega \times V)$ and consequently so is $[M(\lambda - T)^{-1}]^{m+1}$. Moreover $[M(\lambda - T)^{-1}]^{m+2}$ is compact.

Proof:

It remains only to note that Condition (24) means that $\left[\widehat{d\nu}(.)\right] \in L^2(R^n)$ and consequently, by Parseval identity, $d\nu$ is an $L^2(R^n)$-function. It follows that $\left[M(\lambda - T)^{-1}M\right]^m \varphi \in L^2(R^n)$ and this shows that $\left[M(\lambda - T)^{-1}M\right]^m$ and $\left[M(\lambda - T)^{-1}\right]^{m+1}$ are weakly compact in $L^1(\Omega \times V)$. The fact that $M(\lambda - T)^{-1}$ maps weakly compact sets into compact sets ([2] Prop 3) implies that $\left[M(\lambda - T)^{-1}\right]^{m+2}$ is compact. $\square$

We give now a practical condition on $d\mu$ which ensures (24).

Theorem 4 *We suppose there exist $c > 0$ and $\alpha > 0$ such that*

$$\sup_{e \in S^{n-1}} d\mu \left\{v;\ |v.e| \leq \varepsilon\right\} \leq c\varepsilon^{\alpha}$$

then (24) is satisfied for all $m > \frac{n(\alpha+1)}{2\alpha}$.

Proof:

We note that

$$\left|\widehat{d\beta}(\zeta)\right| = \left|\int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}\right| \leq \int_{R^n} \frac{d\mu(v)}{\sqrt{\lambda^2 + |\zeta|^2 |e.v|^2}}$$

where $e = \frac{\zeta}{|\zeta|}$. Hence, for every $\varepsilon > 0$,

$$\begin{aligned} \left|\widehat{d\beta}(\zeta)\right| &\leq \frac{1}{\lambda} d\mu \left\{|e.v| < \varepsilon\right\} + \frac{\|d\mu\|}{|\zeta| \varepsilon} \\ &\leq \left(\frac{1}{\lambda} + \|d\mu\|\right)\left(\varepsilon^{\alpha} + \frac{1}{|\zeta| \varepsilon}\right). \end{aligned}$$

The choice $\varepsilon = \frac{1}{|\zeta|^{\frac{1}{\alpha+1}}}$ leads to $\left|\widehat{d\beta}(\zeta)\right| \leq \frac{\frac{1}{\lambda} + \|d\mu\|}{|\zeta|^{\frac{\alpha}{\alpha+1}}}$ and to

$$\left|\widehat{d\beta}(\zeta)\right|^{2m} \leq \frac{\left(\frac{1}{\lambda} + \|d\mu\|\right)^{2m}}{|\zeta|^{\frac{2m\alpha}{\alpha+1}}}.$$

Hence it suffices that $\frac{2m\alpha}{\alpha+1} > n$, i.e. $m > \frac{n(\alpha+1)}{2\alpha}$. $\square$

5 Model evolution equations with nonincoming boundary conditions

We deal now with the c_0-group $\{V(t); t \in R\}$ generated by $T + M$ where M is the velocity averaging operator (2). As in Section 3, we look for conditions on $d\mu$ under which

$$[MUM]^m = [MUM] * \cdots * [MUM] \quad (m \text{ times})$$

is weakly compact. According to (21)

$$U(t)\varphi \leq RU_\infty(t)E\varphi; \quad \varphi \in L^1_+(\Omega \times V)$$

so that, for $\varphi \in L^1_+(\Omega \times V)$,

$$MU(t)M\varphi \leq MRU_\infty(t)EM\varphi = RMU_\infty(t)ME\varphi$$

from which it follows easily that

$$[MUM]^m \leq R\,[MU_\infty M]^m\, E\varphi.$$

Thus, by a domination argument, it suffices to show that

$$R\,[MU_\infty M]^m : L^1(R^n \times V) \to L^1(\Omega)$$

is weakly compact. To this end, it suffices that $[MU_\infty M]^m$ maps continuously $L^1(R^n \times V)$ *into* $L^2(R^n)$. Indeed, the injection of $L^2(\Omega)$ into $L^1(\Omega)$ is weakly compact because Ω has a *finite Lebesgue measure*. On the other hand,

$$\begin{aligned} MU_\infty(t)M\varphi &= \int (M\varphi)(x - tv)d\mu(v) \\ &= \int (M\varphi)(x - y)d\mu_t(y) \\ &= d\mu_t * M\varphi, \quad \varphi \in L^1(R^n \times V) \end{aligned}$$

where $d\mu_t$ is the image of $d\mu$ under the dilation $v \to tv$. On the other hand, the operator $[MU_\infty(.)M]^2\,(t)$ acts as

$$\varphi \to d\mu_. * M(d\mu_. * M\varphi) = \|d\mu\| \int_0^t d\mu_s * d\mu_{t-s} * M\varphi ds$$

i.e.

$$[MU_\infty(.)M]^2 \varphi = \|d\mu\| \, d\beta^2(t) * M\varphi$$

where

$$d\beta^2(t) = \int_0^t d\mu_s * d\mu_{t-s} ds.$$

By induction,

$$[MU_\infty(.)M]^m \varphi = \|d\mu\|^{m-1} d\beta^m(t) * M\varphi$$

where $d\beta^m(t)$ is defined inductively by

$$d\beta^{j+1}(t) := \int_0^t d\mu_s * d\beta^j(t-s) ds, \ (j \geq 2).$$

By choosing an *even* integer $m = 2p$ ($p \in N$),

$$[MU_\infty(.)M]^{2p} \varphi = \|d\mu\|^{2p-1} d\beta^{2p}(t) * M\varphi$$

and consequently, the L^1 Fourier transform of $[MU_\infty(.)M]^{2p} \varphi$ is equal to

$$\|d\mu\|^{2p-1} \left(\widehat{d\beta^2(t)}(\zeta)\right)^p \widehat{M\varphi}(\zeta).$$

As for the torus, a slightly stronger condition than (14) turns out to be the *key* of the problem:

Theorem 5 *We assume there exist* $0 < \tau < 1$ *and* $m > \frac{n}{(1-\tau)}$ *an even integer such that*

$$\int_{|\zeta|\geq 1} d\zeta \left[\sup_{e \in S^{n-1}} d\mu \otimes d\mu \left\{ (v, v') \in V \times V; \ |(v - v').e| < \frac{1}{|\zeta|^\tau} \right\} \right]^m < \infty \tag{25}$$

Then the remainder terms $R_j(t)$ *are weakly compact for all* $t \geq 0$ *and* $j \geq m+1$. *In particular, if there exist* $c > 0$ *and* $\alpha > 0$ *such that*

$$\sup_{e \in S^{n-1}} d\mu \otimes d\mu \left\{ (v, v') \in V \times V; \ |(v - v').e| < \varepsilon \right\} \leq c\varepsilon^\alpha$$

then $R_j(t)$ *are weakly compact for all* $t \geq 0$ *and* $j > \frac{n(\alpha+1)}{\alpha} + 1$.

Proof: As in Section 3,

$$\begin{aligned}\widehat{d\beta^2(t)}(\zeta) &= \int_0^t \widehat{d\mu_s}(\zeta)\widehat{d\mu_{t-s}}(\zeta)ds \\ &= \int\int \frac{e^{-itv'.\zeta}-e^{-itv.\zeta}}{i(v-v').\zeta}d\mu(v)d\mu(v')\end{aligned}$$

and $\left|\widehat{d\beta^2(t)}(\zeta)\right|$ is majorized by

$$\begin{aligned}& c_t \sup_{e\in S^{n-1}} d\mu\otimes d\mu\left\{(v,v')\in V\times V;\ |(v-v').e|<\frac{1}{|\zeta|^\tau}\right\}+\frac{2\,\|d\mu\|^2}{|\zeta|^{1-\tau}} \\ =\ & c_t a(\zeta)+b(\zeta)\end{aligned}$$

where $c_t = t\sup_{p\neq q}\left|\frac{e^{ip}-e^{iq}}{p-q}\right|$ and $b(\zeta):=\frac{2\|d\mu\|^2}{|\zeta|^{1-\tau}}$. It follows that

$$\begin{aligned}\left|\widehat{d\beta^4(t)}(\zeta)\right| &= \left|\int_0^t \widehat{d\beta^2(s)}(\zeta)\widehat{d\beta^2(t-s)}(\zeta)ds\right| \\ &\leq \int_0^t \left|\widehat{d\beta^2(s)}(\zeta)\right|\left|\widehat{d\beta^2(t-s)}(\zeta)\right|ds \\ &\leq \int_0^t (c_s a(\zeta)+b(\zeta))(c_{t-s}a(\zeta)+b(\zeta))ds \\ &\leq c_t(2)(a(\zeta)+b(\zeta))^2\end{aligned}$$

where $c_t(2)$ is a constant in ζ. It follows, by induction, that

$$\left|\widehat{d\beta^{2p}(t)}(\zeta)\right|\leq c_t(p)(a(\zeta)+b(\zeta))^p$$

where $c_t(p)$ is a constant in ζ. Thus the modulus of the Fourier transform of $[MU_\infty(.)M]^{2p}\varphi$ is majorized by

$$\begin{aligned}& c_t(p)\,\|d\mu\|^{2p-1}\left|\widehat{M\varphi}(\zeta)\right|(a(\zeta)+b(\zeta))^p \\ \leq\ & c_t(p)\,\|d\mu\|^{2p-1}\,\|M\varphi\|_{L^1}(a(\zeta)+b(\zeta))^p.\end{aligned}$$

Hence, knowing that $m=2p$, $[MU_\infty(.)M]^m\varphi$ belongs to $L^2(R^n)$ provided that

$$\int_{|\zeta|\geq 1}(a(\zeta)+b(\zeta))^m d\zeta<\infty$$

and therefore provided that

$$\int_{|\zeta|\geq 1} a(\zeta)^m d\zeta + \int_{|\zeta|\geq 1} b(\zeta)^m d\zeta < \infty.$$

Since $b(\zeta) = \frac{2\|d\mu\|^2}{|\zeta|^{1-\tau}}$, this is possible if

$$\int_{|\zeta|\geq 1} a(\zeta)^m d\zeta < \infty \text{ with } m > \frac{n}{1-\tau} \tag{26}$$

which amounts to our assumption (25). Hence $[MU_\infty(.)M]^m$ is weakly compact and so is $R_{m+1}(t)$. By the convex compactness property of the strong operator topology, it follows that $R_j(t)$ is weakly compact for all $j \geq m+1$. If $\sup_{e\in S^{n-1}} d\mu\left\{v;\ |v.e| \leq \varepsilon\right\} \leq c\varepsilon^\alpha$ then

$$a(\zeta) + b(\zeta) \leq c\frac{1}{|\zeta|^{\alpha\tau}} + \frac{2\left\|d\mu\right\|^2}{|\zeta|^{1-\tau}}.$$

The choice $\alpha\tau = 1-\tau$ (i.e. $\tau = \frac{1}{\alpha+1}$) leads to $a(\zeta)+b(\zeta) \leq \frac{c'}{|\zeta|^{\frac{\alpha}{\alpha+1}}}$ and (25) amounts to $m > \frac{n(\alpha+1)}{\alpha}$. $\square$

As for the torus, the weak compactness of some remainder term $R_m(t)$ for all $t \geq 0$ implies the compactness of $R_{m+2}(t)$ (see [8]).

6 Complementary results

In the present section, we show the *optimality,* in some sense, of the preceeding results. We restrict ourselves to nonincoming boundary conditions.

Theorem 6 *Let $n \geq 3$ and $\Omega \subset R^n$ be a convex open set. Then:*

(i)

$$(\lambda - T + M)^{-1} - (\lambda - T)^{-1}$$

is not weakly compact.

(ii) For all $t > 0$, $V(t) - U(t)$ is not weakly compact.

Proof:

(i) It is easy to see that

$$(\lambda - T + M)^{-1} - (\lambda - T)^{-1} = \sum_{m=1}^{\infty} (\lambda - T)^{-1}\left[M(\lambda - T)^{-1}\right]^m \tag{27}$$

so that

$$(\lambda - T + M)^{-1} \geq (\lambda - T)^{-1} M (\lambda - T)^{-1}$$

in the lattice sense. Hence the weak compactness of $(\lambda - T + M)^{-1} - (\lambda - T)^{-1}$ would imply that $(\lambda - T)^{-1} M (\lambda - T)^{-1}$ is *also* weakly compact. Let us show that the latter is *not* weakly compact if $n \geq 3$. It is easy to see that

$$\begin{aligned}
& (\lambda - T)^{-1} M (\lambda - T)^{-1} f \\
= & \int_0^{\tau(x,v)} e^{-\lambda t} dt \int_V d\mu(v') \int_0^{\tau(x,v')} e^{-\lambda s} f(x - tv - sv', v') ds \\
= & \int_V d\mu(v') \int_0^{\tau(x,v)} \int_0^{\tau(x,v')} e^{-\lambda t} e^{-\lambda s} f(x - tv - sv', v') ds dt \\
= & \int_V d\mu(v') \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\lambda s} f(x - tv - sv', v') ds dt
\end{aligned}$$

where f has been *extended by zero* outside Ω thanks to the convexity of Ω. Let $\{f_j\}_j \subset L^1(\Omega \times V)$ be a normalized sequence converging in the weak star topology of measures to the Dirac mass $\delta_{(0,\overline{v})} = \delta_{x=0} \otimes \delta_{v=\overline{v}}$ where $\overline{v} \in V$. Let $\psi \in C_0(\Omega \times V)$ the space of continuous functions on $\Omega \times V$ tending to zero at the boundary $\partial\Omega$. Then

$$\int_{\Omega \times V} ((\lambda - T)^{-1} M (\lambda - T)^{-1} f_j) \psi$$

is equal to

$$\int_V d\mu(v) \int_\Omega \psi(x, v) dx \int_V d\mu(v') \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\lambda s} f_j(x - tv - sv', v') ds dt$$

or

$$\begin{aligned}
& \int_V d\mu(v) \int_V d\mu(v') \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\lambda s} ds dt \int_\Omega \psi(y + tv + sv', v) f_j(y, v') dy \\
= & \int_{\Omega \times V} dy d\mu(v') f_j(y, v') \left[\int_V d\mu(v) \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\lambda s} \psi(y + tv + sv', v) ds dt \right]
\end{aligned}$$

which tends to

$$\int_V d\mu(v) \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\lambda s} \psi(tv + s\overline{v}, v) ds dt.$$

where ψ has been *extended by zero* outside Ω. Thus $(\lambda-T)^{-1}M(\lambda-T)^{-1}f_j$ converges, in the weak start topology, to the finite Radon measure $d\beta$ on $\Omega\times V$:

$$\psi\in C_0(\Omega\times V)\to\int_V d\mu(v)\int_0^\infty\int_0^\infty e^{-\lambda t}e^{-\lambda s}\psi(tv+s\overline{v},v)dsdt$$

We claim that $d\beta$ is *not* a function. Suppose the contrary, i.e. there exists $f\in L^1(\Omega\times V)$ such that $\forall\psi\in C_0(\Omega\times V)$

$$\int_V d\mu(v)\int_0^\infty\int_0^\infty e^{-\lambda t}e^{-\lambda s}\psi(tv+s\overline{v},v)dsdt\ =\int_{\Omega\times V}f(x,v)\psi(x,v)dxd\mu(v).$$

On the other hand, since for $d\mu$-*almost all* $v\in V$,

$$f(.,v):x\to f(x,v)\quad\in L^1(\Omega)$$

then, for $d\mu$-*almost all* $v\in V$, the measure on Ω

$$\psi\in C(\Omega)\to\int_0^\infty\int_0^\infty e^{-\lambda t}e^{-\lambda s}\psi(tv+s\overline{v})dsdt \tag{28}$$

is equal to *the* L^1 *function* $f(.,v)$, *i.e.* is a density measure

$$\psi\in C(\Omega)\to\int_\Omega f(x,v)\psi(x)dx.$$

This is impossible since the measure (28) is supported on the *bidimensional* linear space spanned by v and $\overline{v}$. This shows that $(\lambda-T)^{-1}M(\lambda-T)^{-1}$ is *not* weakly compact.

(ii) The Dyson-Philips expansion $V(t)-U(t)=\sum_{j=1}^\infty U_j(t)$ shows that $V(t)-U(t)\geq U_1(t)$ in the lattice sense so that the weak compactness of $V(t)-U(t)$ for *some* $t>0$ would imply that $U_1(t)$ is also weakly compact. Let us show that $U_1(t)$ is *not* weakly compact. Note that

$$U_1(t)f=\int_0^t U(t-s)MU(s)fds$$

is equal to

$$\begin{aligned}&\int_0^t ds\int f(x-(t-s)v-sv',v')\times\\&\chi\left\{s<\tau(x-(t-s)v,v')\right\}\chi\left\{t-s<\tau(x,v)\right\}d\mu(v')\\=\ &\int_0^t ds\int f(x-(t-s)v-sv',v')\chi\left\{(s,t);x-(t-s)v-sv'\in\Omega\ \right\}d\mu(v')\end{aligned}$$

where $x \in \Omega$. Let $\psi \in C_0(\Omega \times V)$. Let $\{f_j\}_j \subset L^1(\Omega \times V)$ be a normalized sequence converging in the weak star topology of measures to the Dirac mass $\delta_{(0,\overline{v})} = \delta_{x=0} \otimes \delta_{v=\overline{v}}$ where $\overline{v} \in V$. Then $\langle U_1(t)f_j, \psi\rangle$ is equal to

$$
\begin{aligned}
& \int_{\Omega\times V} dx d\mu(v)\psi(x,v) \int_0^t ds \int f_j(x-(t-s)v - sv', v') \times \\
& \chi\{(s,t); x-(t-s)v-sv' \in \Omega\}\, d\mu(v') \\
= & \int d\mu(v) \int d\mu(v') \int_0^t ds \int_\Omega \psi(x,v) f_j(x-(t-s)v-sv', v') \times \\
& \chi\{(s,t); x-(t-s)v-sv' \in \Omega\}\, dx \\
= & \int d\mu(v) \int d\mu(v') \int_0^t ds \int_\Omega \psi(y+(t-s)v+sv', v) f_j(y, v') \times \\
& \chi\{(s,t); y+(t-s)v+sv' \in \Omega\}\, dy \\
= & \int_{\Omega\times V} f_j(y,v') dx d\mu(v) \int d\mu(v) \int_0^t \psi(y+(t-s)v+sv', v) \times \\
& \chi\{(s,t); y+(t-s)v+sv' \in \Omega\}\, ds
\end{aligned}
$$

and therefore $\{U_1(t)f_j\}$ converges in the weak star topology of measures on $\Omega \times V$ to

$$\psi \in C(\Omega\times V) \to \int d\mu(v) \int_0^t \psi((t-s)v+s\overline{v}, v)\chi\{(s,t); y+(t-s)v+sv' \in \Omega\}\, ds.$$

Let us show that it is *not* a function: Suppose there exists $f \in L^1(\Omega \times V)$ such that

$$
\begin{aligned}
& \int d\mu(v) \int_0^t \psi((t-s)v+s\overline{v}, v)\chi\{(s,t); y+(t-s)v+sv' \in \Omega\}\, ds \\
= & \int_{\Omega\times V} f(x,v)\psi(x,v) dx d\mu(v).
\end{aligned}
$$

Then, for $d\mu$-almost all $v \in V$,

$$\int_0^t \psi((t-s)v+s\overline{v}, v)\chi\{(s,t); y+(t-s)v+sv' \in \Omega\}\, ds = \int_\Omega f(x,v)\psi(x,v) dx$$

and consequently, for $d\mu$-almost all $v \in V$, the Radon measure on Ω

$$\psi \in C(\Omega) \to \int_0^t \psi((t-s)v+s\overline{v})\chi\{(s,t); y+(t-s)v+sv' \in \Omega\}\, ds \quad (29)$$

is an L^1 function, namely $f(.,v)$, and this is not possible since the support of (29) is contained in the bidimensional linear space spanned by v and $\overline{v}$. $\square$

Remark 5 *It is not difficult to show that* $(\lambda-T+M)^{-1}-(\lambda-T)^{-1}$ *is weakly compact if and only if* $(\lambda-T)^{-1}M(\lambda-T)^{-1}$ *is. Thus, Thm 6 shows that we cannot hope to avoid the hypothesis that some "iterate" of* $M(\lambda-T)^{-1}$ *is weakly compact. Similarly,* $V(t)-U(t)$ *is weakly compact if and only if* $U_1(t)$ *is and Thm 6 shows that we cannot avoid to appeal to remainder terms* $R_j(t)$ *with* $j\geq 2$. *This justifies, a posteriori, Vidav's assumptions* [13] [14] *but only for the* L^1 *theory. The situation is completely different in* L^p $(1<p<\infty)$ [9] . *As in Prop 1, we can show that if the hyperplanes have zero* $d\mu$*-measure then* $(\lambda-T+M)^{-1}-(\lambda-T)^{-1}$ *maps weakly compact sets into compact ones. The same result holds for* $V(t)-U(t)$ *if the affine hyperplanes have zero* $d\mu$*-measure* [8].

The case $n=1$ is *quite surprising*. Indeed, we have:

Theorem 7 *Let* $n=1$ *and* $\Omega=\left]-a,a\right[$. *Let* $d\mu$ *be a positive Radon measure on* R *with support* V.

(i) $M(\lambda-T)^{-1}$ *is an integral operator but is not weakly compact.*

(ii) If $d\mu\left\{0\right\}=0$ *then* $(\lambda-T)^{-1}M$ *is a compact (integral) operator and consequently* $(\lambda-T+M)^{-1}-(\lambda-T)^{-1}$ *is compact.*

(iii) We assume that $d\mu$ *is such that* $d\mu\left\{[v-\varepsilon,v+\varepsilon]\right\}\to 0$ *as* $\varepsilon\to 0$ *uniformly in* $v\in V$. *Then* $V(t)-U(t)$ *is weakly compact for all* $t\geq 0$.

Proof:

(i) The fact that $M(\lambda-T)^{-1}$ is not weakly compact has been noted in Prop 1. It is also easy to see that it is an *integral* operator.

(ii) We note that

$$O\varphi \;=\; (\lambda-T)^{-1}M\varphi=\begin{cases}\frac{1}{|v|}\int_{-a}^{x}e^{-\lambda\frac{|x-y|}{|v|}}M\varphi(y)dy \text{ if } v>0\\ \frac{1}{|v|}\int_{x}^{a}e^{-\lambda\frac{|x-y|}{|v|}}M\varphi(y)dy \text{ if } v<0.\end{cases}$$

Let $\left\{h_k\right\}_k$ be a sequence of continuous functions with compact supports such that, for each k, h_k vanishes in some neighborhood of $v=0$ and $h_k\to 1$ in $L^1(V)$ (note that $d\mu$ is finite and $d\mu\left\{0\right\}=0$). We "approximate" O by

$$O_k:\varphi\to\begin{cases}\frac{h_k(v)}{|v|}\int_{-a}^{x}e^{-\lambda\frac{|x-y|}{|v|}}M\varphi(y)dy \text{ if } v>0\\ \frac{h_k(v)}{|v|}\int_{x}^{a}e^{-\lambda\frac{|x-y|}{|v|}}M\varphi(y)dy \text{ if } v<0.\end{cases}$$

It is not difficult to prove that O_k is a *compact* operator in $L^1(\Omega \times V)$. On the other hand,

$$\begin{aligned}
\|O\varphi - O_k\varphi\| &= \int_0^{+\infty} d\mu(v) \int_{-a}^{a} dx \left| \frac{1-h_k(v)}{|v|} \int_{-a}^{x} e^{-\lambda \frac{|x-y|}{|v|}} M\varphi(y)dy \right| \\
&\quad + \int_{-\infty}^{0} d\mu(v) \int_{-a}^{a} dx \left| \frac{1-h_k(v)}{|v|} \int_{x}^{a} e^{-\lambda \frac{|x-y|}{|v|}} M\varphi(y)dy \right| \\
&\le \int_0^{+\infty} d\mu(v) \int_{-a}^{a} dx \frac{|1-h_k(v)|}{|v|} \int_{-a}^{x} e^{-\lambda \frac{|x-y|}{|v|}} M\left|\varphi\right|(y)dy \\
&\quad + \int_{-\infty}^{0} d\mu(v) \int_{-a}^{a} dx \frac{|1-h_k(v)|}{|v|} \int_{x}^{a} e^{-\lambda \frac{|x-y|}{|v|}} M\left|\varphi\right|(y)dy \\
&\le \int_0^{+\infty} d\mu(v) \int_{-\infty}^{\infty} dx \frac{|1-h_k(v)|}{|v|} \int_{-a}^{a} e^{-\lambda \frac{|x-y|}{|v|}} M\left|\varphi\right|(y)dy \\
&\quad + \int_{-\infty}^{0} d\mu(v) \int_{-\infty}^{\infty} dx \frac{|1-h_k(v)|}{|v|} \int_{-a}^{a} e^{-\lambda \frac{|x-y|}{|v|}} M\left|\varphi\right|(y)dy \\
&= \frac{2}{\lambda} \int_0^{+\infty} d\mu(v) \left|1-h_k(v)\right| \int_{-a}^{a} M\left|\varphi\right|(y)dy \\
&\quad + \frac{2}{\lambda} \int_{-\infty}^{0} d\mu(v) \left|1-h_k(v)\right| \int_{-a}^{a} M\left|\varphi\right|(y)dy.
\end{aligned}$$

Hence

$$\|O - O_k\| \le \frac{2\left\|M\right\|_{L(L^1,L^1)}}{\lambda} \left\|1-h_k\right\|_{L^1(V)} \to 0 \; as \; k \to \infty$$

which shows that O is compact.

(iii) We recall that $V(t)-U(t)$ is weakly compact for all $t \ge 0$ *if and only if* $U_1(t)$ is weakly compact for all $t \ge 0$ [7] Chap 2, Thm 2.6. Let us show that $U_1(t)$ *is* weakly compact. We note that $U_1(t)\varphi$ is equal to

$$\begin{aligned}
&\int_0^t ds \int \varphi(x-(t-s)v-sv',v')\chi\left\{(s,t);x-(t-s)v-sv' \in \Omega\right\} d\mu(v') \\
= &\int d\mu(v') \int_0^t \varphi(x-(t-s)v-sv',v')\chi\left\{(s,t);x-(t-s)v-sv' \in \Omega\right\} ds.
\end{aligned}$$

On the other hand, $\chi\left\{(s,t);x-(t-s)v-sv' \in \Omega\right\} = 1$ amounts to

$$x - tv + s(v-v') \in \left]-a,a\right[$$

so

$$\int_0^t \varphi(x-(t-s)v-sv',v')\chi\left\{(s,t);x-(t-s)v-sv'\in\Omega\right\}ds$$
$$= \begin{cases} \int_{(x-tv)\vee(-a)}^{(x-tv')\wedge a}\varphi(y,v')\frac{dy}{|v-v'|} \ if \ \ v'<v \\ \int_{(x-tv')\vee(-a)}^{(x-tv)\wedge a}\varphi(y,v')\frac{dy}{|v-v'|} \ if \ \ v'>v \end{cases}$$

and

$$\begin{aligned} U_1(t)\varphi &= \int_{-\infty}^{v} d\mu(v')\int_{(x-tv)\vee(-a)}^{(x-tv')\wedge a}\varphi(y,v')\frac{dy}{|v-v'|} \\ &\quad + \int_{v}^{\infty} d\mu(v')\int_{(x-tv')\vee(-a)}^{(x-tv)\wedge a}\varphi(y,v')\frac{dy}{|v-v'|} \\ &= O_1\varphi+O_2\varphi. \end{aligned}$$

Let us show that both O_1 and O_2 are weakly compact. We restrict ourselves for instance to O_1 since the same argument holds for O_2. Note that O_1 is an *integral operator*

$$O_1\varphi=\int_V\int_{-a}^{+a}\varphi(y,v')E(v,v',x,y)dyd\mu(v')$$

with kernel

$$E(v,v',x,y):=\chi\left\{v'<v\right\}\chi\left\{y+tv'\leq x\leq y+tv\right\}|v-v'|^{-1} \tag{30}$$

Let

$$O_1^{\varepsilon}:\varphi\to\int_V\int_{-a}^{+a}\varphi(y,v')E_{\varepsilon}(v,v',x,y)dyd\mu(v')$$

with kernel

$$E_{\varepsilon}(v,v',x,y)=E(v,v',x,y)\chi\left\{|v-v'|\geq\varepsilon\right\}.$$

One sees that O_1^{ε} is weakly compact since $E_{\varepsilon}(.,.,.,.)$ is *bounded* and $[-a,a]\times V$ has a finite measure. It suffices to show that $O_1^{\varepsilon}\to O_1$ as $\varepsilon\to 0$ in the *norm operator* topology. We note that $\|O_1\varphi-O_1^{\varepsilon}\varphi\|$ is equal to

$$\int_V d\mu(v)\int_{-a}^{+a}dx\ \int_V\int_{-a}^{+a}|\varphi(y,v')|\,E(v,v',x,y)\chi\left\{|v-v'|<\varepsilon\right\}dyd\mu(v')$$
$$=\int_V d\mu(v')\int_{-a}^{+a}|\varphi(y,v')|\,dy\int_V\chi\left\{|v-v'|<\varepsilon\right\}d\mu(v)\int_{-a}^{+a}E(v,v',x,y)dx.$$

On the other hand, (30) shows that

$$\int_{-a}^{+a} E(v,v',x,y)dx \leq |v-v'|^{-1} \int_{y+tv'}^{y+tv} dx = t$$

whence

$$\begin{aligned} \|O_1\varphi - O_1^\varepsilon\varphi\| &\leq t\int_V d\mu(v') \int_{-a}^{+a} |\varphi(y,v')|\, dy \int_V \chi\left\{|v-v'|<\varepsilon\right\} d\mu(v) \\ &\leq t \sup_{v'\in V} d\mu\left\{[v'-\varepsilon, v'+\varepsilon]\right\} \|\varphi\| \end{aligned}$$

and

$$\|O_1 - O_1^\varepsilon\| \leq t \sup_{v'\in V} d\mu\left\{[v'-\varepsilon, v'+\varepsilon]\right\} \to 0 \text{ as } \varepsilon \to 0.$$

Remark 6 *(i) Note that the assumption* $\sup_{v'\in V} d\mu\left\{[v'-\varepsilon, v'+\varepsilon]\right\} \to 0$ *as* $\varepsilon \to 0$ *is satisfied by the Lebesgue measure on* R.

(ii) If V *is bounded then* $\sup_{v'\in V} d\mu\left\{[v'-\varepsilon, v'+\varepsilon]\right\} \to 0$ *as* $\varepsilon \to 0$ *is equivalent to the assumption that* $d\mu$ *is diffuse, i.e.* $d\mu\left\{v'\right\} = 0$ *for all* $v' \in V$.

(iii) The (weak) compactness of $(\lambda - T)^{-1}K$ *in one dimension has already been proved in* [6] *for general collision operator* K.

(iv) The case $n = 2$ *is a limiting case between the two different situations described in Thm 6 and Thm 7. However we conjecture the plausible result:*

Conjecture 1 *Thm 6 is still true for* $n = 2$.

Remark 7 *Thm 6 (ii) solves in the positive (for* $n \geq 3$*) a conjecture by the author* [7] *Chap 4. This conjecture turned out to be false in* L^p $(1 < p < \infty)$ *(see* [9]*).*

7 Applications to spectral theory

In this section, we show how the above compactness results provide a sound foundation to the L^1 spectral theory. We restrict ourselves to nonincoming boundary conditions but the same results hold on the torus. Let $\Omega \subset R^n$ be an arbitrary open set with *finite* Lebesgue measure and $d\mu$ be a positive (*not* necessarily finite) Radon measure on R^n with support V. Let K be a *collision* operator

$$K : \varphi \in L^1(\Omega \times V) \to \int_V k(x,v,v')\varphi(x,v')d\mu(v') \in L^1(\Omega \times V)$$

with the natural assumption

$$\int_V |k(.,v,.)|\, d\mu(v) \in L^\infty(\Omega \times V).$$

Let $\left\{V^K(t); t \geq 0\right\}$ the c_0-semigroup generated by $T+K$. Following B. Lods [5], we suppose that K is *regular* in L^1 in the sense that the family of operators (indexed by $x \in \Omega$)

$$\psi \in L^1(V) \to \int_V k(x,v,v')\varphi(v')d\mu(v') \in L^1(V)$$

is *collectively weakly compact*. This amounts to

$$\{|k(x,.,v')|\,;\ (x,v') \in \Omega \times V\} \text{ is relatively weakly compact} \tag{31}$$

in $L^1(V)$. This assumption can be checked by the well-known Dunford-Pettis criterion (see [1]). We note that the *positive* collision operator

$$|K| : \varphi \in L^1(\Omega \times V) \to \int_V |k(x,v,v')|\, \varphi(x,v')d\mu(v') \in L^1(\Omega \times V)$$

is *also* regular. On the other hand,

$$\left|\left[K(\lambda - T)^{-1}\right]^m \varphi\right| \leq \left[|K|\,(\lambda - T)^{-1}\right]^m |\varphi|$$

and

$$\left|U_j^K(t)\varphi\right| \leq U_j^{|K|}(t)\,|\varphi|$$

where $\left\{U_j^K\right\}$ denotes the terms of the Dyson-Philips expansion of $V^K(t)$ and $\left\{U_j^{|K|}\right\}$ those of the semigroup $V^{|K|}(t)$ generated by $T+|K|$. Thus, as far as the weak compactness is concerned, by using domination arguments, there is no loss of generality to assume that the collision operator K is *positive*. On the other hand, if $k_i(x,v,v') = k(x,v,v')\chi_{\{v\in V;|v|\leq i\}}$ and

$$K_i\varphi = \int_V k_i(x,v,v')\varphi(x,v')d\mu(v')$$

then

$$\begin{aligned}\|K\varphi - K_i\varphi\| &\leq \int_{\Omega\times\{v\in V;|v|>i\}}\int_V k(x,v,v')\,|\varphi(x,v')|\,d\mu(v') \\ &\leq \sup_{(x,v')\in\Omega\times V}\int_{\{v\in V;|v|>i\}} k(x,v,v')d\mu(v)\,\|\varphi\|_{L^1(\Omega\times V)}\end{aligned}$$

and, by (31),

$$\|K\varphi - K_i\varphi\| \leq \sup_{(x,v')\in\Omega\times V}\int_{\{v\in V;|v|>i\}} k(x,v,v')d\mu(v) \to 0 \text{ as } i\to\infty.$$

Thus, we may replace K by some *truncation* K_i since $\left[K(\lambda-T)^{-1}\right]^m$ and $U_j^K(t)$ depends continuously on K in the norm operator topology . This means that we may suppose without loss of generality that V is bounded and consequently that $d\mu$ is *finite*. A basic property of a positive *regular* collision operator is that it can be approximated in the norm operator topology by collision operators *dominated* by collision operators of the form

$$\varphi\in L^1(\Omega\times V)\to f(v)\int_V \varphi(x,v')d\mu(v') \tag{32}$$

where $f\in L^1(V)$ [5] . Thus we may assume that K has the form (32). By approximation again we may suppose that $f\in L^1(V)\cap L^\infty(V)$ and finally, by a domination argument, we may even assume that f is a *constant* c. In such a case, the collision operator K is nothing but the velocity *averaging* operator

$$M:\varphi\in L^1(\Omega\times V)\to c\int_V \varphi(x,v')d\mu(v').$$

Hence, the following compactness results are simple consequences of Thm 3, Thm 4 and Thm 5.

Theorem 8 *Let $\Omega\subset R^n$ $(n\geq 2)$ be an arbitrary open set with finite Lebesgue measure. Let $d\mu$ be a positive (not necessarily finite) Radon measure on R^n and K be a regular collision operator in the sense* (31).

(i) We assume that for all $c>0$ there exist $c'>0$ and $\alpha>0$ such that

$$\sup_{e\in S^{n-1}} d\mu\{v;|v|\leq c,\ |v.e|\leq\varepsilon\}\leq c'\varepsilon^\alpha. \tag{33}$$

Then some power of $K(\lambda-T)^{-1}$ is weakly compact.

(ii) We assume that for all $c > 0$ there exist $c' > 0$ and $\alpha > 0$ such that

$$\sup_{e \in S^{n-1}} d\mu \otimes d\mu \{(v, v'); |v| \leq c, |v'| \leq c, \ |(v - v').e| < \varepsilon\} \leq c' \varepsilon^{\alpha}. \tag{34}$$

Then some remainder term of the Dyson-Philips expansion is weakly compact.

Remark 8 *In general, the advection semigroup $U(t)$ contains an absorption term, i.e., has the form:*

$$U(t)\varphi = e^{-\int_0^t \sigma(x - sv, v)ds} \varphi(x - tv, v) \chi_{\{t \leq \tau(x,v)\}}$$

where $\sigma(.,.) \in L^{\infty}(\Omega \times V)$ (or at least bounded below) is the collision frequency. Mathematically speaking, this does not add any difficulty since, by domination arguments, we may assume that $\sigma(.,.)$ is a constant. Thus Thm 8 above remains true.

Remark 9 *For $n = 1$, we have more precise results since Thm 7 remains true for regular collision operators.*

We are ready to summarize the spectral results:

Theorem 9 *Let $\Omega \subset R^n$ be an arbitrary open set with finite Lebesgue measure. Let $d\mu$ be a positive (not necessarily finite) Radon measure on R^n and K be a regular collision operator in the sense* (31).

(i) Let $n \geq 2$. If (33) *is satisfied then $\sigma(T + K) \cap \{\mathrm{Re}\,\lambda > s(T)\}$ consists of at most isolated eigenvalues with finite algebraic multiplicities where $s(T)$ is the spectral bound of T. If* (34) *is satisfied then $\{U(t); t \geq 0\}$ and $\{V(t); t \geq 0\}$ have the same essential type and consequently, in the region $\left\{\nu; |\nu| > e^{s(T)t}\right\}$, $\sigma(V(t))$ consists of at most isolated eigenvalues with finite algebraic multiplicities.*

(ii) Let $n = 1$. If $d\mu \{0\} = 0$ then $\sigma(T + K) \cap \{\mathrm{Re}\,\lambda > s(T)\}$ consists of at most isolated eigenvalues with finite algebraic multiplicities. If $\sup_{v' \in V} d\mu \{[v' - \varepsilon, v' + \varepsilon]\} \to 0$ as $\varepsilon \to 0$ then $\sigma(V(t)) \cap \left\{\nu; |\nu| > e^{s(T)t}\right\}$ consists of at most isolated eigenvalues with finite algebraic multiplicities.

Remark 10 *Apart from the one dimensional case where, thanks to Thm 7, we can appeal to the stability of the essential spectrum by weakly compact perturbation* [4]*, the analysis of $\sigma(T + K) \cap \{\mathrm{Re}\,\lambda \leq s(T)\}$ and $\sigma(V(t)) \cap \left\{\nu; |\nu| \leq e^{s(T)t}\right\}$ for $n \geq 2$ relies on different tools* [10].

8 On L^1 "averaging lemmas"

We know that in all dimensions $M(\lambda - T)^{-1}$ is *never* (weakly) compact [2] Example 1 or Prop 1 (i) above. It may be of interest to look for practical bounded subsets of $L^1(\Omega\times V)$ which are mapped by $M(\lambda-T)^{-1}$ into (weakly) compact sets. We will restrict ourselves to *nonincoming boundary conditions*.

Theorem 10 *Let $n = 1$ and $\Omega = \left]-a, a\right[$. Let $d\mu$ be a positive Radon measure on R such that $d\mu\{0\} = 0$. If $\Xi \subset L^1(\Omega \times V)$ is a bounded subset such that*

$$\int_{-\varepsilon}^{\varepsilon} d\mu(v) \int_{-a}^{a} |\varphi(y, v)|\, dy \to 0 \ \ as\ \varepsilon \to 0 \tag{35}$$

uniformly in $\varphi \in \Xi$, then $\{M(\lambda - T)^{-1}\varphi;\ \varphi \in \Xi\}$ is relatively compact in $L^1(\Omega)$.

Proof:
A simple calculation shows that

$$(\lambda - T)^{-1}\varphi = \begin{cases} \frac{1}{|v|}\int_{-a}^{x} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy \text{ if } v > 0 \\ \frac{1}{|v|}\int_{x}^{a} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy \text{ if } v < 0 \end{cases}$$

so that

$$\begin{aligned} M(\lambda - T)^{-1}\varphi &= \int_0^\infty d\mu(v)\frac{1}{|v|}\int_{-a}^{x} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy \\ &+ \int_{-\infty}^{0} d\mu(v)\frac{1}{|v|}\int_{x}^{a} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy = O\varphi. \end{aligned}$$

Let O_ε the truncated operator

$$\varphi \to \int_\varepsilon^\infty d\mu(v)\frac{1}{|v|}\int_{-a}^{x} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy + \int_{-\infty}^{-\varepsilon} d\mu(v)\frac{1}{|v|}\int_{x}^{a} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy.$$

A simple calculation shows that O_ε is a *compact operator* on $L^1(\Omega \times V)$. On the other hand

$$O\varphi - O_\varepsilon\varphi = \int_{-\varepsilon}^{\varepsilon} d\mu(v)\frac{1}{|v|}\int_{-a}^{x} e^{-\lambda\frac{|x-y|}{|v|}}\varphi(y, v)dy$$

and

$$\begin{aligned}
\|O\varphi - O_\varepsilon\varphi\| &\leq \int_{-\infty}^{\infty} dx \int_{-\varepsilon}^{\varepsilon} d\mu(v) \frac{1}{|v|} \int_{-a}^{a} e^{-\lambda \frac{|x-y|}{|v|}} |\varphi(y,v)| \, dy \\
&\leq \int_{-\varepsilon}^{\varepsilon} d\mu(v) \frac{1}{|v|} \int_{-a}^{a} \left[\int_{-\infty}^{\infty} e^{-\lambda \frac{|x-y|}{|v|}} dx \right] |\varphi(y,v)| \, dy \\
&= \int_{-\varepsilon}^{\varepsilon} d\mu(v) \frac{1}{|v|} \int_{-a}^{a} \left[\int_{-\infty}^{\infty} e^{-\lambda \frac{|z|}{|v|}} dz \right] |\varphi(y,v)| \, dy \\
&= \frac{2}{\lambda} \int_{-\varepsilon}^{\varepsilon} d\mu(v) \int_{-a}^{a} |\varphi(y,v)| \, dy.
\end{aligned}$$

Hence, by (35), $\|O\varphi - O_\varepsilon\varphi\| \to 0$ *as* $\varepsilon \to 0$ uniformly in $\varphi \in \Xi$. This shows that

$$\{O\varphi;\ \varphi \in \Xi\} = \{O_\varepsilon\varphi + (O\varphi - O_\varepsilon\varphi);\ \varphi \in \Xi\}\ \forall \varepsilon > 0$$

is relatively compact in $L^1(\Omega)$. $\square$

Remark 11 *This result improves* [2] *Lemma 8, where it is assumed that* $d\mu\{[-\varepsilon, \varepsilon]\} \leq c\varepsilon^\gamma$ *and that* Ξ *is a bounded subset of* $L^p[d\mu(v);\ L^1(dx)]$ *for some* $p > 1$.

Remark 12 *It is clear that the same arguments used in the proof of Thm 7 (ii) show also that* $(\lambda - \widetilde{T})^{-1}M$ *is compact in* L^1 *where* $\widetilde{T}\varphi = v.\frac{\partial \varphi}{\partial x}$ *and*

$$D(\widetilde{T}) = \left\{ \varphi \in L^1; v.\frac{\partial \varphi}{\partial x} \in L^1,\ \varphi_{|\Gamma_+} = 0 \right\}$$

so that, by duality, we obtain an averaging lemma in $L^\infty(\Omega \times V)$:

Theorem 11 *Let* $n = 1$ *and* $\Omega = \left]-a, a\right[$. *Let* $d\mu$ *be a positive Radon measure on* R *such that* $d\mu\{0\} = 0$. *Then* $M(\lambda - T)^{-1} : L^\infty(\Omega \times V) \to L^\infty(\Omega)$ *is compact.*

Remark 13 *This result complements Lemma 7 in* [2] *where a stronger (Hölder) regularity for velocity averages is obtained under the stronger assumption that* $d\mu\{[-\varepsilon, \varepsilon]\} \leq c\varepsilon^\gamma$.

We extend now Thm 10 to *arbitrary* dimensions under a *stronger* assumption.

Theorem 12 *Let $\Omega \subset R^n$ $(n \geq 2)$ be a bounded and convex open subset and $V = R^n$ endowed with the Lebesgue measure. Let*

$$M : \varphi \in L^1(\Omega \times R^n; dx \otimes dv) \to \int_{R^n} \varphi(x,v)dv \in L^1(\Omega).$$

Let $\Xi \subset L^1(\Omega \times R^n)$ be a bounded subset. We assume that Ξ is "equicontinuous with respect to velocities" in the sense

$$\int_{\Omega \times R^n_v} |\varphi(y, v+z) - \varphi(y,v)|\, dydv \to 0 \tag{36}$$

as $z \to 0$ uniformly in $\varphi \in \Xi$. Then $\{M(\lambda - T)^{-1}\varphi; \varphi \in \Xi\}$ is relatively compact in $L^1(\Omega)$.

Proof:

We note that

$$(\lambda - T)^{-1}\varphi = \int_0^\infty e^{-\lambda t}\varphi(x - tv, v)dt, \quad (x \in \Omega)$$

where φ has been *extended by zero* to R^n_x with respect to the space variable. Moreover,

$$M(\lambda - T)^{-1}\varphi \;\; = \;\; \int_{R^n} dv \int_0^\infty e^{-\lambda t}\varphi(x - tv, v)dt$$

and a simple change of variable yield

$$\psi = M(\lambda - T)^{-1}\varphi = \int_0^\infty e^{-\lambda t}\frac{dt}{t^n}\int_{R^n}\varphi(y, \frac{x-y}{t})dy.$$

It suffices to show that $\int |\psi(x+z) - \psi(x)|\, dx \to 0$ *uniformly* in $\varphi \in \Xi$ as $z \to 0$. We note that

$$\begin{aligned}
\int |\psi(x+z) - \psi(x)|\, dx &\leq \int_0^\infty e^{-\lambda t}\frac{dt}{t^n}\int_{R^n} dy \int \left|\varphi(y, \frac{x+z-y}{t}) - \varphi(y, \frac{x-y}{t})\right| dx \\
&= \int_0^\infty e^{-\lambda t}dt \int_{R^n} dy \int \left|\varphi(y, v + \frac{z}{t}) - \varphi(y,v)\right| dv \\
&= \int_0^\varepsilon e^{-\lambda t}dt \int_{R^n} dy \int \left|\varphi(y, v + \frac{z}{t}) - \varphi(y,v)\right| dv \\
&\quad + \int_\varepsilon^\infty e^{-\lambda t}dt \int_{R^n} dy \int \left|\varphi(y, v + \frac{z}{t}) - \varphi(y,v)\right| dv \\
&\leq 2\varepsilon \left\|\varphi\right\|_{L^1} + \int_\varepsilon^\infty e^{-\lambda t}dt \int_{R^n} dy \int \left|\varphi(y, v + \frac{z}{t}) - \varphi(y,v)\right| dv.
\end{aligned}$$

On the other hand, by assumption, there exists $\alpha > 0$ such that

$$\int_{R^n} dy \int \left|\varphi(y, v + \frac{z}{t}) - \varphi(y, v)\right| dv \leq \varepsilon$$

uniformly in $\varphi \in \Xi$ if $\left|\frac{z}{t}\right| \leq \alpha$. This is true for *all* $t \geq \varepsilon$ if $|z| \leq \varepsilon\alpha$ and consequently

$$\int |\psi(x+z) - \psi(x)|\, dx \leq 2\varepsilon \left\|\varphi\right\|_{L^1} + \lambda^{-1}\varepsilon$$

and the proof is complete. □

Remark 14 *A result in the same spirit and with a different proof appeared recently* [3] *under a weaker assumption : The set Ξ is assumed to satisfy only some "equiintegrability" with respect to velocities. However, the proof is quite involved. On the other hand, arguing as in the proof of Thm 12, we can derive a weak compactness result when Ξ is only "equiintegrable" with respect to velocities. Indeed:*

Definition 1 *A bounded subset of $L^1(R^n_x \times R^n_v; dx \otimes dv)$ is said to be "equiintegrable" with respect to velocities if for each $\varepsilon > 0$ there exists $\alpha > 0$ such that for each measurable familly $(A_y)_{y \in R^n}$ of measurable subsets of R^n satisfying $|A_y| \leq \alpha$ we have $\int dy \int_{A_y} |\varphi(y,v)|\, dv \leq \varepsilon$ uniformly in $\varphi \in \Xi$ where $|A_y|$ is the Lebesgue measure of A_y.*

Theorem 13 *Let $\Omega \subset R^n$ $(n \geq 2)$ be a bounded and convex open subset and $V = R^n$ endowed with the Lebesgue measure. Let $\Xi \subset L^1(\Omega \times R^n)$ be bounded and "equiintegrable" with respect to velocities. Then $\{M(\lambda - T)^{-1}\varphi; \varphi \in \Xi\}$ is relatively weakly compact in $L^1(\Omega)$.*

Proof:
We start as in the proof of Thm 12. We have

$$\psi = M(\lambda - T)^{-1}\varphi = \int_0^\infty e^{-\lambda t} \frac{dt}{t^n} \int_{R^n} \varphi(y, \frac{x-y}{t}) dy.$$

It remains to prove that $\int_A |\psi(x)|\, dx \to 0$ as $|A| \to 0$ *uniformly* in $\varphi \in \Xi$. We note that

$$\begin{aligned}
\int_A |\psi(x)|\, dx &\leq \int_0^\infty e^{-\lambda t} \frac{dt}{t^n} \int_{R^n} dy \int_A \left| \varphi(y, \frac{x-y}{t}) \right| dx \\
&= \int_0^\infty e^{-\lambda t} dt \int_{R^n} dy \int_{\frac{A-y}{t}} |\varphi(y,v)|\, dv \\
&= \int_0^\varepsilon e^{-\lambda t} dt \int_{R^n} dy \int_{\frac{A-y}{t}} |\varphi(y,v)|\, dv \\
&\quad + \int_\varepsilon^\infty e^{-\lambda t} dt \int_{R^n} dy \int_{\frac{A-y}{t}} |\varphi(y,v)|\, dv \\
&\leq \varepsilon \left\| \varphi \right\|_{L^1} + \int_\varepsilon^\infty e^{-\lambda t} dt \int_{R^n} dy \int_{\frac{A-y}{t}} |\varphi(y,v)|\, dv.
\end{aligned}$$

On the other hand

$$\left| \frac{A-y}{t} \right| = \frac{1}{t^n} |A-y| = \frac{1}{t^n} |A| \leq \frac{1}{\varepsilon^n} |A| \quad (t \geq \varepsilon)$$

and the *"equiintegrability"* with respect to velocities show that

$$\int_{R^n} dy \int_{\frac{A-y}{t}} |\varphi(y,v)|\, dv \leq \varepsilon$$

uniformly in $\varphi \in \Xi$ and in $t \geq \varepsilon$ if $|A|$ is small enough. It follows that

$$\int_A |\psi(x)|\, dx \leq \varepsilon \left\| \varphi \right\|_{L^1} + \lambda^{-1}\varepsilon \leq (c + \lambda^{-1})\varepsilon$$

uniformly in $\varphi \in \Xi$ if $|A|$ is small enough and the proof is complete. □

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